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Permutations with Limited Repetitions

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ABSTRACT

An explicit formula in terms of Bell polynomials is derived for the number of r-permutations (called "variations, r at a time" in the older literature) with limited repetition, where each one of the n different things to be permuted may appear at most s times.

1. INTRODUCTION

The classical problem of r-permutations (called "variations, r at a time" in the older literature) with or without repetition respresents only two extreme cases of a more general problem (generally not to be found in the literature), that of r-permutations with limited (or restricted) repetition, where each one of the n different things to be permuted may appear at most s times. One might also speak of r-permutations for objects of type (s^n); or of words of length r, formed from an alphabet of n letters, no letter being allowed to appear in a word more than s times. We shall use the notation $P(n, r, s)$ for the number of these r-permutations (or words) with limited repetition.

To be more precise, there is really a restriction only if $s < r$; for if $s \geq r$ it is obvious that the r-permutations with limited repetition coincide with the classical r-permutations with (unlimited) repetition whose number is n^r; hence:

$$P(n, r, s) = n^r \quad \text{if} \quad s \geq r. \tag{1.1}$$

So the only interesting case is that of $s < r$, and the main purpose of the present paper is to derive for this case the following explicit formula for $P(n, r, s)$ in terms of Bell polynomials, thus furnishing one more instance for the utmost importance of these polynomials[1] in many problems of combinatorial analysis:

$$P(n, r, s) = A_r((n)_1, (n)_2, \ldots, (n)_r; \underbrace{1, 1, \ldots, 1}_{s}, \underbrace{0, 0, \ldots, 0}_{r-s}); \quad (1.2)$$

this means that $P(n, r, s)$ may be derived from the r-th Bell polynomial $A_r(f_1, f_2, \ldots, f_r; g_1, g_2, \ldots, g_r)$ by giving its variables the following values:

$$f_i = (n)_i = n(n-1)(n-2)\cdots(n-i+1) \text{ for } i = 1,2,\ldots, r; \quad (1.3)$$

$$g_i = \begin{cases} 1 \text{ for } i = 1, 2, \ldots, s; \\ 0 \text{ for } i = s+1, s+2, \ldots, r. \end{cases} \quad (1.4)$$

For the sake of completeness it should be mentioned that a similar formula holds if $s \geq r$, only replacing (1.4) with

$$g_i = 1 \text{ for } i = 1, 2, \ldots, r; \quad (1.5)$$

in other words:

$$P(n, r, s) = A_r((n)_1, (n)_2, \ldots, (n)_r; 1, 1, \ldots, 1) \quad \text{if } s \geq r. \quad (1.6)$$

It is indeed easy to see that the right-hand side of (1.6) has the value n^r—in accordance with (1.1)—since in general

$$A_r(f_1, f_2, \ldots, f_r; 1, 1, \ldots, 1) = \sum_{i=1}^{r} S(r, i) f_i \quad (1.7)$$

where the $S(r, i)$ are Stirling numbers of the second kind, defined by

$$n^r = \sum_{i=1}^{r} S(r, i)\,(n)_i. \quad (1.8)$$

Thus in the case $s \geq r$, (1.6) would be nothing but another way of writing the classical formula (1.1).

For this reason our attention will be confined to the case $s < r$. In Sections 2 and 3, two proofs of the formula (1.2) are given for this case

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[1] For their definition see [2, § 2.8], or formula (2.7) below.

alone, although by modifying details one could obtain new proofs of (1.6).

To give an example of the application of (1.2) let $r = 4$, $s = 2$;[2] from (2.7) below or a table of Bell polynomials (see [2, p. 49]) it is seen that

$$A_4 = f_1 g_4 + f_2(4g_3 g_1 + 3g_2^2) + f_3 \cdot 6g_2 g_1^2 + f_4 g_1^4 \,; \tag{1.9}$$

hence by (1.3) and (1.4):

$$P(n, 4, 2) = 3(n)_2 + 6(n)_3 + (n)_4; \tag{1.10}$$

thus, e.g.,

$$P(7, 4, 2) = 3 \cdot 7 \cdot 6 + 6 \cdot 7 \cdot 6 \cdot 5 + 7 \cdot 6 \cdot 5 \cdot 4 = 2226.$$

Generalizing the example (1.10) it is easy to see that (due to the fact that the Bell polynomials are linear in the f_i) formula (1.2) might be interpreted also as an expansion of $P(n, r, s)$ into descending factorials $(n)_1, (n)_2, \ldots, (n)_r$, with coefficients generally depending on r and s. [The extreme case $s \geq r$ where the coefficients are Stirling numbers has already been mentioned; the other extreme case is $s = 1$ where the expansion reduces to the classical formula

$$P(n, r, 1) = (n)_r \tag{1.11}$$

for r-permutations without repetition. Of course the formula (1.2) holds also in the trivial case $r > ns$ where $P(n, r, s) = 0$.]

Finally the author wishes to remark that the analogous problem of combinations with limited repetitions will be considered in another paper.

2. First Proof of Formula (1.2)

It is known—see the remark after formula (12) on p. 11 in [2]—that

$$\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^s}{s!}\right)^n = \sum_{r=0}^{\infty} P(n, r, s) t^r / r!; \tag{2.1}$$

[2] For the case $s = 2$, recurrence formulae for the numbers $P(n, r, 2)$ called—$q_{n,r}$ by Riordan—can be found in [2, problem 13, p. 17]. Also for $s > 2$, recurrence formulae for $P(n, r, s)$ might be obtained from the generating function (2.1), but they remain outside the scope of this paper.

hence by expanding the left-hand side (using the multinomial theorem) and equating coefficients of $t^r/r!$ one obtains the following explicit formula for the number $P(n, r, s)$ of r-permutations with limited repetition:

$$P(n, r, s) = \sum \frac{n!r!}{k_0!k_1!k_2! \cdots k_s!(1!)^{k_1}(2!)^{k_2} \cdots (s!)^{k_s}} \tag{2.2}$$

with the sum over all solutions in nonnegative integers of

$$k_0 + k_1 + k_2 + \cdots + k_s = n \tag{2.3}$$

and

$$k_1 + 2k_2 + 3k_3 + \cdots + sk_s = r. \tag{2.4}$$

But (2.3) allows us to eliminate the factor $k_0!$ from the denominator in the sum on the right-hand side of (2.2), by replacing $n!/k_0!$ with

$$\frac{n!}{[n-(k_1+k_2+\cdots+k_s)]!} =$$
$$n(n-1)(n-2)\cdots[n-(k_1+k_2+\cdots+k_s)+1] = (n)_{k_1+k_2+\cdots+k_s};$$

the formula (2.2) thus becomes

$$P(n, r, s) = \sum \frac{r!(n)_{k_1+k_2+\cdots+k_s}}{k_1!k_2! \cdots k_s!(1!)^{k_1}(2!)^{k_2} \cdots (s!)^{k_s}} \tag{2.5}$$

the sum now being over the non-negative solutions of (2.4) alone.

At a first glance it might seem from (2.3) that the inequality

$$k_1 + k_2 + \cdots + k_s \leq n \tag{2.6}$$

also must be satisfied, and therefore that solutions of (2.4) not fulfilling (2.6) should be excluded from the sum in (2.5); but it turns out that the inclusion of such terms does no harm since the factor $(n)_{k_1+k_2+\cdots+k_s}$ equals zero if $k_1 + k_2 + \cdots + k_s > n$, and so the condition (2.6) will be disregarded in what follows.

On the other hand we have Faà di Bruno's explicit formula for the r-th Bell polynomial (see (45a) on p. 36 of [2]):

$$A_r(f_1, f_2, \ldots, f_r; g_1, g_2, \ldots, g_r)$$
$$= \sum \frac{r!f_{k_1+k_2+\cdots+k_r}}{k_1!k_2!\cdots k_r!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_r}{r!}\right)^{k_r} \tag{2.7}$$

where the sum is over all solutions in non-negative integers of

$$k_1 + 2k_2 + 3k_3 + \cdots + rk_r = r. \tag{2.8}$$

If we now choose (remember that we are only considering the case $s < r$)

$$g_1 = g_2 = \cdots = g_s = 1, \; g_{s+1} = g_{s+2} = \cdots = g_r = 0$$

according to (1.4), those terms of the sum in (2.7) where at least one of the exponents $k_{s+1}, k_{s+2}, \ldots, k_r$ is different from 0, drop out; hence we shall have:

$$A_r(f_1, \ldots, f_r; \underbrace{1, 1, \ldots, 1}_{s}, \underbrace{0, 0, \ldots, 0}_{r-s}) = \sum \frac{r! f_{k_1+k_2+\cdots+k_s}}{k_1! k_2! \cdots k_s! (1!)^{k_1} \cdots (s!)^{k_s}} \tag{2.9}$$

where (2.8) now reduces to (2.4), so that the sum in (2.9) will be over the non-negative solutions of (2.4). Formula (1.2) is now proved by comparison of (2.5) with the special case of formula (2.9) that obtains by letting, according to (1.3), $f_i = (n)_i$ for $i = 1, 2, \ldots, r$.

3. Second Proof of Formula (1.2)

The following proof seems to have the advantage of casting more light on the reasons why the Bell polynomials make their appearance in formula (1.2).

Let us consider any r-permutation $(a_1 a_2 \cdots a_r)$ where the a_i are letters taken from an alphabet A_n of n letters; we might also say that

$$(a_1 a_2 \cdots a_r) = (f(1), f(2), \ldots, f(r)) \tag{3.1}$$

with f: $S_r \to A_n$, $S_r = \{1, 2, \ldots, r\}$.

Now to any r-permutation belongs a *nucleus*, i.e., a partition of S_r into disjoint parts or "blocks" where any two numbers i and j of S_r belong to the same block iff

$$f(i) = f(j); \tag{3.2}$$

and the r-permutation (3.1) will be *admissible* iff no block of the nucleus contains more than s elements. (Here we have called an r-permutation admissible if it is one of the $P(n, r, s)$ r-permutations in which no letter of A_n appears more than s times.)

In general the correspondance between r-permutation and nucleus will not be one-one. Indeed let $B_1, B_2, \ldots, B_\nu$ be the blocks of a nucleus (with no block containing more than s elements); then for any number $i \in B_1, f(i)$ might be any letter from A_n; once that letter is chosen, then, for any $j \in B_2, f(j)$ might again be any letter from A_n different from the preceding one, etc. Hence by a classical argument (ν-permutations without repetition) we see that to a given nucleus (with ν blocks and no block containing more than s elements) belong exactly

$$(n)_\nu = n(n-1)(n-2)\cdots(n-\nu+1) \tag{3.3}$$

admissible r-permutations, where

$$\nu = k_1 + k_2 + \cdots + k_r \tag{3.4}$$

for a nucleus of type $[k_1, k_2, \ldots, k_r]$, i.e., with k_i blocks of i elements $(i = 1, 2, \ldots, r)$.

Confining our attention again to the more interesting case $s < r$, it is evident that the admissible r-permutations are just those for whose nucleus

$$k_{s+1} = k_{s+2} = \cdots = k_r = 0; \tag{3.5}$$

hence we shall have

$$P(n, r, s) = \Sigma(n)_\nu = \Sigma(n)_{k_1+k_2+\cdots+k_r} \tag{3.6}$$

with the sum over those partitions of $S_r = \{1, 2, \ldots, r\}$ whose type satisfies (3.5)—so that we might also write:

$$P(n, r, s) = \Sigma(n)_{k_1+k_2+\cdots+k_s}\,. \tag{3.7}$$

It is now easy to express this sum over the partitions satisfying (3.5) in terms of Bell polynomials. It has been shown by Frucht and Rota [1], using purely combinatorial arguments, that

$$A_r(1, 1, \ldots, 1;\ g_1, g_2, \ldots, g_r) = \Sigma\, g_1^{k_1} g_2^{k_2} \cdots g_r^{k_r} \tag{3.8}$$

with the sum over all partitions of S_r of any type $[k_1, k_2, \ldots, k_r]$;[3]

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[3] In [1] the notation $Y_n(g_1, g_2, \ldots, g_n)$ has been used for the n-th Bell polynomial with $f_1 = f_2 \cdots = f_n = 1$.

multiplying here each term of the sum by a factor of $f_{k_1+k_2+\cdots+k_r}$, the following generalization of (3.8) immediately obtains:

$$A_r(f_1, f_2, \ldots, f_r; g_1, g_2, \ldots, g_r) = \sum f_{k_1+k_2+\cdots+k_r}\, g_1^{k_1}\, g_2^{k_2} \cdots g_r^{k_r}, \quad (3.9)$$

the sum being as before over the partitions of S_r. For the choice (1.4) terms which do not satisfy (3.5) drop out, and we have:

$$A_r(f_1, f_2, \ldots, f_r; \underbrace{1, 1, \ldots, 1}_{s}, \underbrace{0, 0, \ldots, 0}_{r-s}) = \sum f_{k_1+k_2+\cdots+k_r} \quad (3.10)$$
$$= \sum f_{k_1+k_2+\cdots+k_s}$$

with the sums now restricted to those partitions of S_r whose type satisfies (3.5).

It is now obvious that the right-hand sides of (3.6) or (3.7), and (3.10), become equal if we finally put $f_i = (n)_i$ in accordance with (1.3), thus proving again the formula (1.2).

References

1. R. Frucht and G. C. Rota, Polinomios de Bell y particiones de conjuntos finitos, *Scientia* (*Valparaiso*), **126** (1965), 5–10.

2. J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.